Shimura varieties II: examples and types

Ravi Fernando – fernando@berkeley.edu

March 9, 2016*

Write outline on board, and spend a minute or two giving an introduction.

1 Siegel modular variety

Motivation: at the beginning of the semester, Ian told us about complex abelian varieties, polarizations, and a moduli space for principally polarized abelian varieties. In this section, we will look at essentially the same moduli space, but as a Shimura variety. We will start by briefly discussing the dictionary between complex structures and hodge structures of type $(-1,0),(0,-1)$ on a real vector space. Then, using this vocabulary, we will construct a canonical Shimura datum (G,X) associated to a symplectic space (V,ψ), and we will define the Siegel modular variety to be the corresponding Shimura variety $\mathrm{Sh}(G,X)$. Finally, we will discuss the moduli interpretation of this variety.

1.1 Dictionary

We will soon be discussing complex structures (on real vector spaces) and hodge structures (equivalently, representations of the Deligne torus $\mathbb{S}$) of type $(-1,0),(0,-1)$, so let's start by discussing the equivalences between these ideas. Namely, if V is a vector space over $\mathbb{R}$:

$$\{\text{complex structures } J \text{ on } V\} \tag{1}$$
$$\leftrightarrow \{\text{hodge structures on } V \text{ of type } (-1,0),(0,-1)\} \tag{2}$$
$$\leftrightarrow \{\text{representations of } \mathbb{S} \text{ on } V \text{ whose complexifications contain only the characters } z,\overline{z}\} \tag{3}$$

The equivalences are as follows:

$$J \in \mathrm{End}(V), J^2 = -1 \tag{4}$$
$$\to V_{\mathbb{C}} = V^{-1,0} \oplus V^{0,-1} = (i\text{-eigenspace of } J) \oplus (-i\text{-eigenspace}) \tag{5}$$
$$\leftrightarrow h : \mathbb{S} \to \mathrm{GL}(V), z \in \mathbb{S} \text{ acting on } V^{p,q} \subset V_{\mathbb{C}} \text{ by } z^{-p}\overline{z}^{-q} \tag{6}$$
$$\to J = h(i). \tag{7}$$

(The convention that z acts on $V^{p,q}$ by $z^{-p}\overline{z}^{-q}$ is due to Deligne. An easier way to define h_J: $z = a + bi$ acts as $a + bJ$.)

*Notes for a talk given in Berkeley's number theory seminar, organized by Xinyi Yuan and Ken Ribet. Main reference: Milne, *Introduction to Shimura varieties*.

1.2 Symplectic spaces and the associated Shimura data

Let V be a $2n$-dimensional vector space over a field k, which we assume to have characteristic $\neq 2$. If V is equipped with a non-degenerate alternating form ψ, then we call (V, ψ) a *symplectic space*. It can be shown that with respect to some basis $e_{\pm i}$, ψ has the matrix $\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$.

Given a symplectic space (V, ψ), we define $\mathrm{GSp}(\psi)$ to be the group of automorphisms of V preserving ψ up to a scalar:

$$\mathrm{GSp}(\psi) = \{g \in \mathrm{GL}(V) : \psi(gu, gv) = \nu(g) \cdot \psi(u, v) \text{ for some } \nu(g) \in k^\times\} \tag{8}$$

We let $\mathrm{Sp}(\psi)$ be the subgroup of $\mathrm{GSp}(\psi)$ that preserves ψ exactly; i.e. where $\nu(g) = 1$. So we have a short exact sequence $1 \to \mathrm{Sp}(\psi) \to \mathrm{GSp}(\psi) \xrightarrow{\nu} \mathbb{G}_m \to 1$. Some facts: $\mathrm{GSp}(\psi)$ is a reductive group with derived group $\mathrm{Sp}(\psi)$, center $\mathbb{G}_m$, and adjoint group $\mathrm{GSp}(\psi)/\mathbb{G}_m = \mathrm{Sp}(\psi)/\{\pm 1\}$.

For example, in the modular curve case ($n = 1$, $\psi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$), then $\mathrm{GSp}(\psi) = \mathrm{GL}_2$. This is because for any $g \in \mathrm{GL}(V)$, $\psi(gu, gv)$ is a nondegenerate alternating form, and every such form is a scalar multiple of ψ itself. One can calculate that $\psi(gu, gv) = \det(g) \cdot \psi(u, v)$, so that $\mathrm{Sp}(\psi) = \mathrm{SL}_2$. In this case, the Shimura datum we construct will be $(\mathrm{GL}_2, \mathcal{H}^\pm)$, and the resulting Siegel modular varieties will be modular curves.

Fix a symplectic space (V, ψ) over $\mathbb{Q}$. We want to define an associated Shimura datum (G, X). We set $G = \mathrm{GSp}(\psi)$, and $S = \mathrm{Sp}(\psi)$. (Not to be confused with the Deligne torus $\mathbb{S}$.) Recall that X is supposed to be a $G(\mathbb{R})$-conjugacy class of homomorphisms $h : \mathbb{S} \to G_\mathbb{R}$ satisfying certain properties. But by the dictionary from before, we will equivalently define X as a $G_\mathbb{R}$-conjugacy class of complex structures J whose corresponding homomorphisms $h_J : \mathbb{S} \to \mathrm{GL}(V_\mathbb{R})$ have images contained in $\mathrm{GSp}(\psi)_\mathbb{R}$.

What does it mean for h_J to have its image contained in $\mathrm{GSp}(\psi)_\mathbb{R}$? It means that $\psi(zu, zv)$ is a constant multiple of $\psi(u, v)$ for all $z = a + bi \in \mathbb{S}$. A straightforward calculation implies that this happens if and only if we have $\psi(Ju, Jv) = \psi(u, v)$, and when this happens, we get $\psi(zu, zv) = |z|^2 \psi(u, v)$. Moreover, all of these conditions are equivalent to saying that the form $\psi_J(u, v) := \psi(u, Jv)$ is a symmetric form. To summarize:

$$h = h_J : \mathbb{S} \to G_\mathbb{R} \tag{9}$$
$$\leftrightarrow J \text{ with } \psi(zu, zv) = |z|^2 \psi(u, v) \tag{10}$$
$$\leftrightarrow J \text{ with } \psi(Ju, Jv) = \psi(u, v) \tag{11}$$
$$\leftrightarrow J \text{ with } \psi_J(u, v) := \psi(u, Jv) \text{ symmetric.} \tag{12}$$

Let J be a complex structure satisfying these equivalent conditions. We say J is *positive* or *negative* if $\psi_J(u, v)$ is a positive (respectively negative) definite form on $V_\mathbb{R}$. We let X^+ and X^- denote the sets of positive and negative complex structures, respectively, and set $X = X^+ \cup X^-$.

Connection to the classical discussion: in Ian's talk, we started with a complex torus $\mathbb{C}^g / \Lambda$. Our question was essentially this: for which Λ can we endow $\mathbb{C}^g$ with a Riemann form, i.e. a positive definite Hermitian form whose imaginary part is integral on Λ? Here, we are starting with a real torus $\mathbb{R}^{2g} / \mathbb{Z}^{2g} = V_{\mathbb{R}}/\Lambda$ equipped with an alternating form ψ that is already integral on the lattice, and our question is: for which complex structures J (i.e. identifications of $\mathbb{R}^{2g}$ with $\mathbb{C}^g$, with $J =$ multiplication by i) is the form $\psi_J - i\psi$ a Riemann form? Answer: the ones in X^+.

We claim that X gives a conjugacy class of maps $h_J : \mathbb{S} \to G_{\mathbb{R}}$, and it satisfies the three Shimura datum axioms. First, there are several easy things to check: $G(\mathbb{R})$ acts on X by $g \cdot J = gJg^{-1}$, $S(\mathbb{R}) = \mathrm{Sp}(\psi)(\mathbb{R})$ acts transitively on X^+ (essentially because it acts transitively on symplectic bases), and an element $g \in G(\mathbb{R})$ preserves X^+ and X^- if $\nu(g) > 0$, and switches them otherwise. As a consequence of these, $G(\mathbb{R})$ acts transitively by conjugation on X, so it is exactly one conjugacy class, as desired.

Now we will sketch a proof that (G, X) satisfies the three axioms of a Shimura datum, namely:

1. For all (equivalently some) $h \in X$, only the weights $(-1, 1), (0, 0)$, and $(1, -1)$ occur in the representation of $\mathbb{S}$ on $\mathrm{Lie}(G^{\mathrm{ad}})_{\mathbb{C}}$ induced by h.
2. $\mathrm{ad}\, h(i) = \mathrm{ad}\, J$ is a Cartan involution on $G^{\mathrm{ad}}_{\mathbb{R}}$.
3. G^{ad} has no $\mathbb{Q}$-factor on which the projection of h is trivial.

Proofs:

1. Fix $h \in X$, and observe that $\mathrm{Lie}(G)_{\mathbb{C}} \subset \mathrm{Lie}(\mathrm{GL}(V))_{\mathbb{C}} = \mathrm{End}(V_{\mathbb{C}}) = \mathrm{Hom}(V_{\mathbb{C}}, V_{\mathbb{C}})$. Here, $\mathbb{S}$ acts by $h(z)^{-1}$ on the source of the Hom, and by $h(z)$ on the target. We claim that only the weights $(-1, 1), (0, 0)$, and $(1, -1)$ appear in even the larger representation. To prove this, recall that $V_{\mathbb{C}} = V^{-1,0} \oplus V^{0,-1}$, where z acts as z on $V^{-1,0}$ and as $\overline{z}$ on $V^{0,-1}$. We observe that $\mathrm{Hom}(V_{\mathbb{C}}, V_{\mathbb{C}})$ then decomposes into a direct sum of four pieces, and z acts on $\mathrm{Hom}(V^{p,q}, V^{p',q'})$ as $z^{p-p'}\overline{z}^{q-q'}$. So the weights we get are all possible differences among $(-1, 0)$ and $(0, -1)$, which are exactly $(-1, 1), (0, 0)$, and $(1, -1)$.
2. Since $\mathrm{ad}\, J^2 = \mathrm{ad}(-1)$ is trivial, $\mathrm{ad}\, J$ is an involution. To prove that it is Cartan, we need to show that the locus $\{g \in \mathbb{C} : G = \mathrm{ad}\, J(\overline{g})\}$ is compact, but in fact it is contained in the unitary group $U(\psi) = U_g$.
3. $G^{\mathrm{ad}} = G/\mathbb{G}_m$ is simple over $\mathbb{Q}$, and h acts nontrivially on it.

Aside: there are three more axioms for Shimura data that one sometimes wants in addition to these three. I will ignore these in my talk, but they are also easy to prove for the data we have constructed.

Let's return to the modular curve case for a moment. Here, $G = \mathrm{GSp}_2 = \mathrm{GL}_2$ (over $\mathbb{Q}$), and X consists of the $\mathrm{GL}_2(\mathbb{R})$-conjugacy class of the matrix $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We claim $X = \mathbb{C} \setminus \mathbb{R}$ as a manifold. One way to see this is to note that any matrix in X has eigenvalues $\pm i$, and the corresponding eigenvectors (defined up to multiplicative constants) are non-real and conjugate to each other. This allows us to choose $\begin{bmatrix} 1 \\ \tau \end{bmatrix}$ as an eigenvector for i, $\tau \in \mathbb{C} \setminus \mathbb{R}$, and its conjugate as an eigenvector for $-i$.

1.3 The Siegel modular variety and moduli interpretation

Let (V, ψ) be a symplectic space, and (G, X) the resulting Shimura datum. The *Siegel modular variety* attached to (V, ψ) is the Shimura variety $\mathrm{Sh}(G, X)$.

Proposition 1.1. *The set* $\mathrm{Sh}_K(G, X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$ *is in bijection with the set* $\mathcal{M}_K$ *of triples* $(A, s, \eta K)$ *where:*

- A *is an object in the isogeny category of abelian varieties;*
- $\pm s$ *is a polarization for* $\Lambda = H_1(A, \mathbb{Q})$*; and*
- ηK *is a* K*-orbit of isomorphisms* $V(\mathbb{A}_f) \to H_1(A, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{A}_f$ *sending* ψ *to an* $\mathbb{A}_f^\times$*-multiple of* s*,*

modulo isomorphisms of these data.

2 Various types of Shimura varieties

In this section, we will discuss three types of Shimura varieties. They are named after their respective moduli interpretations: PEL Shimura varieties parametrize abelian varieties equipped with polarization, endomorphism, and level structure; Shimura varieties of hodge type parametrize abelian varieties equipped with hodge tensors (as well as the data from before); and Shimura varieties of abelian type parametrize abelian motives equipped with some extra structure.

The implications among them are: PEL $\implies$ hodge type $\implies$ abelian type.

2.1 Hodge type

A Shimura datum (G, X) is said to be of *hodge type* if it embeds into one of the Shimura data we have just discussed; i.e. if there exists a symplectic space (V, ψ) over $\mathbb{Q}$ and an injective homomorphism $\rho : G \to G(\psi)$ carrying X into $X(\psi)$.

2.2 Abelian type

Recall that to any Shimura datum (G, X), there is a corresponding connected Shimura datum (G^{der}, X^+). In particular, in the Siegel modular variety case, we have a connected Shimura datum $(S(\psi), X^+(\psi))$, where $S(\psi) = \mathrm{GSp}(\psi)^{\mathrm{der}} = \mathrm{Sp}(\psi)$. With this in mind, we give a series of definitions:

A connected Shimura datum (H, X^+), with H simple, is of *primitive abelian type* if it embeds into the connected Shimura datum $(S(\psi), X(\psi)^+)$ defined by a symplectic space (V, ψ).

A connected Shimura datum (H, X^+) is of *abelian type* if there exist Shimura data (H_i, X_i^+) of primitive abelian type and an isogeny $\prod_i H_i \to H$ carrying $\prod_i X_i^+$ into X^+.

An arbitrary Shimura datum (G, X) is of *abelian type* if the corresponding connected Shimura datum (G^{der}, X^+) is of abelian type. A Shimura variety is of *abelian type* if it is attached to a Shimura datum of abelian type.

Deligne has given a classification of Shimura data of abelian type. If (G, X^+) is a connected Shimura datum with G simple, then whether it is of abelian type depends on where in the classification of simple Lie groups. If G^{ad} is of type A, B, or C, it is of abelian type; if E_6 or E_7, it is not; if D, it may or may not be. (What about $\mathrm{E}_8, \mathrm{F}_4, \mathrm{G}_2$?)

2.3 PEL Shimura varieties

Let k be a field of characteristic 0. As with the previous types, PEL Shimura varieties will be characterized by their Shimura data, but in order to define PEL data we first need to study k-algebras with involution. In this section, a k-algebra will be a (not necessarily commutative) ring containing k in its center, and finite-dimensional over k.

An *involution* of a k-algebra B is a k-linear map $* : B \to B$ with $b^{**} = b$ and $(ab)^* = b^*a^*$. It follows that $*$ acts trivially on $k \subset B$.

Proposition 2.1. *Suppose k is algebraically closed (of characteristic 0), and let $(B, *)$ be a semisimple k-algebra with involution. Then $(B, *)$ is isomorphic to a product of pairs of the following types:*

1. *(A) $M_n(k) \times M_n(k)$ with $(a, b)^* = (b^T, a^t)$;*
2. *(C) $M_n(k)$ with $b^* = b^T$;*
3. *(BD) $M_n(k)$ with $b^* = Jb^TJ^{-1}$, where n is even and $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$*

To define PEL data, we will only be interested in the A and C types. The reason is that the A, C, and BD types will respectively correspond to groups whose derived subgroups are (twists of powers of) SL, Sp, and SO; we want to avoid the SO case because $G \approx \mathrm{O}$ isn't connected.

We will now roughly mimic the construction of the data that gave us Siegel modular varieties, but carrying more information around. If $(B, *)$ is a semisimple k-algebra with involution, we say (V, ψ) is a *symplectic* $(B, *)$*-module* if V is a B-module and $\psi : V \times V \to k$ is a nondegenerate alternating k-bilinear form with $\psi(bu, v) = \psi(u, b^*v)$ for all $b \in B$ and $u, v \in V$.

Some assumptions: say F is the center of B and F_0 consists of the $*$-invariants of F. We assume that B and V are both free over F, and that for all homomorphisms $\rho : F_0 \to \overline{k}$ over k, the algebras $(B \otimes_{F_0,\rho} \overline{k}, *)$ are of the same type, either A, C, or BD. (It is sufficient but not necessary for F to be a field.) Then we define $G \subset \mathrm{GL}(V)$ to be the group

$$G(\mathbb{Q}) = \{g \in \mathrm{Aut}_B(V) : \psi(gu, gv) = \mu(g)\psi(u, v) \text{ for some } \mu(g) \in k^\times\}. \tag{13}$$

We further let $G' = \ker(\mu) \cap \ker(\det)$, where the determinant makes sense in $\mathrm{GL}(V)$ (over k) rather than over B.

Let C be a semisimple $\mathbb{R}$-algebra with an involution $*$. We say $*$ is *positive* if $\mathrm{Tr}_{C/\mathbb{R}}(c^*c) > 0$ for all nonzero $c \in C$. (The trace here is the trace of the multiplication-by-(c^*c) map on the finite-dimensional $\mathbb{R}$-vector space C. There are some other conditions equivalent to this.)

We are now ready to define PEL data. Let B be a simple $\mathbb{Q}$-algebra with a positive involution $*$ (i.e. $*$ becomes positive after base-change to $\mathbb{R}$), and let (V, ψ) be a symplectic $(B, *)$-module. Assume that $(B, *)$ is of type A or C, and choose G as above.

Proposition 2.2. *There is a unique $G(\mathbb{R})$ conjugacy class X of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$ such that each $h \in X$ defines a complex structure on $V(\mathbb{R})$ that is positive or negative for ψ. The pair (G, X) satisfies the three axioms of a Shimura datum, as well as additional axiom 4 (the weight is rational).*

The Shimura data arising from this process are called *simple PEL data* of type A or C.